

# Macdonald polynomials at $t = q^k$

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## Abstract

We investigate the homogeneous symmetric Macdonald polynomials  $P_\lambda(\mathbb{X}; q, t)$  for the specialization  $t = q^k$ . We show an identity relying the polynomials  $P_\lambda(\mathbb{X}; q, q^k)$  and  $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ . As a consequence, we describe an operator whose eigenvalues characterize the polynomials  $P_\lambda(\mathbb{X}; q, q^k)$ .

## 1 Introduction

Macdonald polynomials are  $(q, t)$ -deformations of Schur functions which play an important rôle in the representation theory of the double affine Hecke algebra [10, 12] since they are the eigenfunctions of the Cherednik elements. The polynomials considered here are the homogeneous symmetric Macdonald polynomials  $P_\lambda(\mathbb{X}; q, t)$  and are the eigenfunctions of the Sekiguchi-Debiard operator. For  $(q, t)$  generic, these polynomials are completely characterized by their eigenvalues, since the dimensions of the eigenspaces is 1. It is no longer the case when  $t$  is specialized to a rational power of  $q$ . Hence, it is more convenient to characterize the Macdonald (homogeneous symmetric) polynomials by orthogonality (*w.r.t.* a  $(q, t)$ -deformation of the usual scalar product on symmetric functions) and by some conditions on their dominant monomials (see *e.g.* [11]). In this paper, we consider the specialization  $t = q^k$  where  $k$  is a strictly positive integer. One of our motivations is to generalize an identity of [1], which shows that even powers of the discriminant are rectangular Jack polynomials. Here, we show that this property follows from deeper relations between the Macdonald polynomials  $P_\lambda(\mathbb{X}; q, q^k)$  and  $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$  (in the  $\lambda$ -ring notation). This result is interesting in the

context of the quantum fractional Hall effect [7], since it implies properties of the expansion of the powers of the discriminant in the Schur basis [3, 5, 13]. It implies also that the Macdonald polynomials (for  $t = q^k$ ) are characterized by the eigenvalues of an operator  $\mathfrak{M}$  whose eigenspaces are of dimension 1 described in terms of isobaric divided differences.

The paper is organized as follow. After recalling notations and background (Section 2) for Macdonald polynomials, we give, in Section 3, some properties of the operator which substitutes a complete function to each power of a letter. These properties allow to show our main result in Section 4 which is an identity relying the polynomial  $P_\lambda(\mathbb{X}; q, q^k)$  and  $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ . As a consequence, we describe (Section 5) an operator  $\mathfrak{M}$  whose eigenvalues characterize the Macdonald polynomials  $P_\lambda(\mathbb{X}; q, q^k)$ . Finally, in Section 6, we give an expression of  $\mathfrak{M}$  in terms of Cherednik elements.

## 2 Notations and background

Consider an alphabet  $\mathbb{X}$  potentially infinite. We will use the notations of [9] for the generating function  $\sigma_z(\mathbb{X})$  of the complete homogeneous functions  $S^p(\mathbb{X})$ ,

$$\sigma_z(\mathbb{X}) = \sum_i S^i(\mathbb{X}) z^i = \prod_i \frac{1}{1 - xz}.$$

The algebra  $Sym$  of symmetric function has a structure of  $\lambda$ -ring [9]. We recall that the sum of two alphabets  $\mathbb{X} + \mathbb{Y}$  is defined by

$$\sigma_z(\mathbb{X} + \mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S^i(\mathbb{X} + \mathbb{Y}) z^i.$$

In particular, if  $\mathbb{X} = \mathbb{Y}$  one has  $\sigma_z(2\mathbb{X}) = \sigma_z(\mathbb{X})^2$ . This definition is extended for any complex number  $\alpha$  by  $\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha$ . For example, the generating series of the elementary function is

$$\begin{aligned} \lambda_z(\mathbb{X}) &:= \sum \Lambda_i(\mathbb{X}) z^i = \prod_x (1 + xz) \\ &= \sigma_{-z}(-\mathbb{X}) = \sum_i (-1)^i S^i(-\mathbb{X}) z^i. \end{aligned}$$

The complete functions of the product of two alphabets  $\mathbb{X}\mathbb{Y}$  are given by the Cauchy kernel

$$K(\mathbb{X}, \mathbb{Y}) := \sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S^i(\mathbb{X}\mathbb{Y}) = \prod_{x \in \mathbb{X}} \prod_{y \in \mathbb{Y}} \frac{1}{1 - xyt} = \sum_\lambda S_\lambda(\mathbb{X}) S_\lambda(\mathbb{Y}),$$

where  $S_\lambda$  denotes, as in [9], a Schur function. More generally, one has

$$K(\mathbb{X}, \mathbb{Y}) = \sum_{\lambda} A_{\lambda}(\mathbb{X}) B_{\lambda}(\mathbb{Y})$$

for any pair of basis  $(A_{\lambda})_{\lambda}$  and  $(B_{\lambda})_{\lambda}$  in duality for the usual scalar product  $\langle \cdot, \cdot \rangle$ .

## 2.1 Macdonald polynomials

One considers the  $(q, t)$ -deformation (see *e.g.* [11]) of the usual scalar product on symmetric functions defined for a pair of power sum functions  $\Psi^{\lambda}$  and  $\Psi^{\mu}$  (in the notation of [9]) by

$$\langle \Psi^{\lambda}, \Psi^{\mu} \rangle_{q,t} = \delta_{\lambda,\mu} z_{\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad (1)$$

where  $\delta_{\lambda,\mu} = 1$  if  $\lambda = \mu$  and 0 otherwise. The family of Macdonald polynomials  $(P_{\lambda}(\mathbb{X}; q, t))_{\lambda}$  is the unique basis of symmetric functions orthogonal for  $\langle \cdot, \cdot \rangle_{q,t}$  verifying

$$P_{\lambda}(\mathbb{X}; q, t) = m_{\lambda}(\mathbb{X}) + \sum_{\mu \leq \lambda} u_{\lambda\mu} m_{\mu}(\mathbb{X}), \quad (2)$$

where  $m_{\lambda}$  denotes, as usual, a monomial function [9, 11]. The reproducing kernel associated to this scalar product is

$$K_{q,t}(\mathbb{X}, \mathbb{Y}) := \sum_{\lambda} \langle \Psi^{\lambda}, \Psi^{\lambda} \rangle_{q,t}^{-1} \Psi_{\lambda}(\mathbb{X}) \Psi_{\lambda}(\mathbb{Y}) = \sigma_1 \left( \frac{1-t}{1-q} \mathbb{X} \mathbb{Y} \right)$$

see *e.g.* [11] (VI. 2). In particular, one has

$$K_{q,t}(\mathbb{X}, \mathbb{Y}) = \sum_{\lambda} P_{\lambda}(\mathbb{X}; q, t) Q_{\lambda}(\mathbb{Y}; q, t), \quad (3)$$

where  $Q_{\lambda}(\mathbb{X}; q, t)$  is the dual basis of  $P_{\lambda}(\mathbb{Y}; q, t)$  for  $\langle \cdot, \cdot \rangle_{q,t}$ ,

$$Q_{\lambda}(\mathbb{X}; q, t) = \langle P_{\lambda}, P_{\lambda} \rangle_{q,t}^{-1} P_{\lambda}(\mathbb{X}; q, t). \quad (4)$$

The coefficient  $b_\lambda(q, t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1}$  is known to be

$$b_\lambda(q, t) = \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_j - i + 1} t^{\lambda'_i - j}}{1 - q^{\lambda_j - i} t^{\lambda'_i - j + 1}} \quad (5)$$

see [11] VI.6. Writing

$$K_{q,t} \left( \left( \frac{1-q}{1-t} \right) \mathbb{X}, \mathbb{Y} \right) = K(\mathbb{X}, \mathbb{Y}), \quad (6)$$

one finds that  $(P_\lambda \left( \left( \frac{1-q}{1-t} \right) \mathbb{X}; q, t \right))_\lambda$  is the dual basis of  $(Q_\lambda(\mathbb{X}; q, t))_\lambda$  for the usual scalar product  $\langle \cdot, \cdot \rangle$ .

Note that there exists an other Kernel type formula which reads

$$\lambda_1(\mathbb{X}\mathbb{Y}) = \sum_{\lambda} P_{\lambda'}(\mathbb{X}; t, q) P_\lambda(\mathbb{Y}; q, t) = \sum_{\lambda} Q_{\lambda'}(\mathbb{X}; t, q) Q_\lambda(\mathbb{Y}; q, t). \quad (7)$$

where  $\lambda'$  denotes the conjugate partition of  $\lambda$ . This formula can be found in [11] VI.5 p 329.

From Equalities (6) and (3), one has

$$\sigma_1(\mathbb{X}\mathbb{Y}) = K_{q,t} \left( \frac{1-q}{1-t} \mathbb{X}, \mathbb{Y} \right) = \sum_{\lambda} Q_\lambda \left( \frac{1-q}{1-t} \mathbb{X}; q, t \right) P_\lambda(\mathbb{Y}; q, t). \quad (8)$$

Applying (7) to

$$\sigma_1(\mathbb{X}\mathbb{Y}) = \lambda_{-1}(-\mathbb{X}\mathbb{Y}),$$

one obtains

$$\sigma_1(\mathbb{X}\mathbb{Y}) = \sum_{\lambda} (-1)^{|\lambda|} Q_{\lambda'}(-\mathbb{X}; t, q) Q_\lambda(\mathbb{Y}; q, t). \quad (9)$$

Identifying the coefficient of  $P_\lambda(\mathbb{Y}; t, q)$  in (8) and (9), one finds the property below.

**Lemma 2.1**

$$Q_\lambda(-\mathbb{X}; t, q) = (-1)^{|\lambda|} P_{\lambda'} \left( \frac{1-q}{1-t} \mathbb{X}; q, t \right). \quad (10)$$

Unlike the usual ( $q = t = 1$ ) scalar product, there is no expression as a constant term for the product  $\langle, \rangle_{q,t}$  when  $\mathbb{X} = \{x_1, \dots, x_n\}$  is finite. But the Macdonald polynomials are orthogonal for an other scalar product defined by

$$\langle f, g \rangle'_{q,t;n} = \frac{1}{n!} \text{C.T.} \{f(\mathbb{X})g(\mathbb{X}^\vee)\Delta_{q,t}(\mathbb{X})\} \quad (11)$$

where C.T. denotes constant term *w.r.t.* the alphabet  $\mathbb{X}$ ,  $\Delta_{q,t}(\mathbb{X}) = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty}$ ,  $(a; b)_\infty = \prod_{i \geq 0} (1 - ab^i)$  and  $\mathbb{X}^\vee = \{x_1^{-1}, \dots, x_n^{-1}\}$ . The expression of  $\langle P_\lambda, Q_\lambda \rangle'_{q,t;n}$  is given by ([11] VI.9)

$$\langle P_\lambda, Q_\lambda \rangle'_{q,t;n} = \frac{1}{n!} \text{C.T.} \{\Delta_{q,t}(\mathbb{X})\} \prod_{(i,j) \in \lambda} \frac{1 - q^{i-1} t^{n-j+1}}{1 - q^i t^{n-j}}. \quad (12)$$

## 2.2 Skew symmetric functions

Let us define as in [11] VI 7, the skew  $Q$  functions by

$$\langle Q_{\lambda/\mu}, P_\nu \rangle_{q,t} := \langle Q_\lambda, P_\mu P_\nu \rangle_{q,t}. \quad (13)$$

Straightforwardly, one has

$$Q_{\lambda/\mu}(\mathbb{X}; q, t) = \sum_\nu \langle Q_\lambda, P_\nu P_\mu \rangle_{q,t} Q_\nu(\mathbb{X}; q, t). \quad (14)$$

And classically, the following property hold. <sup>1</sup>

**Proposition 2.2** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two alphabets, one has*

$$Q_\lambda(\mathbb{X} + \mathbb{Y}; q, t) = \sum_\mu Q_\mu(\mathbb{X}; q, t) Q_{\lambda/\mu}(\mathbb{Y}; q, t),$$

*or equivalently*

$$P_\lambda(\mathbb{X} + \mathbb{Y}; q, t) = \sum_\mu P_\mu(\mathbb{X}; q, t) P_{\lambda/\mu}(\mathbb{Y}; q, t).$$

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<sup>1</sup>See *e.g.* [11] VI.7 for a short proof of this identity

Equalities (3) and (7) are generalized by identities 15 and 16 as shown in [11] example 6 p.352

$$\sum_{\rho} P_{\rho/\lambda}(\mathbb{X}; q, t) Q_{\rho/\mu}(\mathbb{Y}; q, t) = K_{qt}(\mathbb{X}, \mathbb{Y}) \sum_{\rho} P_{\mu/\rho}(\mathbb{X}; q, t) Q_{\lambda/\rho}(\mathbb{Y}; q, t), \quad (15)$$

$$\sum_{\rho} Q_{\rho'/\lambda'}(\mathbb{X}; t, q) Q_{\rho/\mu}(\mathbb{Y}; q, t) = \lambda_1(\mathbb{X}\mathbb{Y}) \sum_{\rho} Q_{\mu'/\rho'}(\mathbb{X}, t, q) Q_{\lambda/\rho}(\mathbb{Y}; q, t). \quad (16)$$

### 3 The substitution $x^p \rightarrow S^p(\mathbb{Y})$ and the Macdonald polynomials

Let  $\mathbb{X} = \{x_1, \dots, x_n\}$  be a finite alphabet and  $\mathbb{Y}$  be an other (potentially infinite) alphabet. For simplicity we will denote by  $\int_{\mathbb{Y}}$  the substitution

$$\int_{\mathbb{Y}} x^p = S^p(\mathbb{Y}), \quad (17)$$

for each  $x \in \mathbb{X}$  and each  $p \in \mathbb{Z}$ .

#### 3.1 Substitution formula

Let us define the symmetric function

$$\mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}; q, t) := \frac{1}{n!} \int_{\mathbb{Y}} P_{\lambda}(\mathbb{X}; q, t) Q_{\mu}(\mathbb{X}^{\vee}; q, t) \Delta(\mathbb{X}, q, t) \quad (18)$$

where  $\mathbb{X}^{\vee} = \{x_1^{-1}, \dots, x_n^{-1}\}$ .

Set  $\mathbb{Y}^{tq} := \frac{1-t}{1-q} \mathbb{Y}$  and consider the substitution

$$\int_{\mathbb{Y}^{tq}} x^p = S^p(\mathbb{Y}^{tq}) = Q_p(\mathbb{Y}; q, t). \quad (19)$$

One has the following property.

**Theorem 3.1** *Let  $\mathbb{X} = \{x_1, \dots, x_n\}$  be an alphabet,  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition and  $\mu \subset \lambda$ . The polynomial  $\mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}^{tq}; q, t)$  is the Macdonald polynomial*

$$\mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}^{tq}; q, t) = \frac{1}{n!} \prod_{(i,j) \in \lambda} \frac{1 - q^{i-1} t^{n-j+1}}{1 - q^i t^{n-j}} \text{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{\lambda/\mu}(\mathbb{Y}, q, t) \quad (20)$$

**Proof** From the definition of the  $Q_\lambda$ , one has

$$\int_{\mathbb{Y}^{tq}} x^p = Q_p(\mathbb{Y}; q, t) = \text{C.T.}\{x^{-p} K_{q,t}(x, \mathbb{Y})\}. \quad (21)$$

Hence, the polynomial  $\mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}^{qt}, q, t)$  is the constant term

$$\mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}^{tq}; q, t) = \frac{1}{n!} \text{C.T.}\{P_\lambda(\mathbb{X}^\vee; q, t) Q_\mu(\mathbb{X}; q, t) K_{q,t}(\mathbb{X}, \mathbb{Y}) \Delta(\mathbb{X}, q, t)\}.$$

As a special case of Equality (15),

$$K_{qt}(\mathbb{X}, \mathbb{Y}) Q_\mu(\mathbb{X}; q, t) = \sum_{\rho} P_{\rho/\mu}(\mathbb{Y}; q, t) Q_\rho(\mathbb{X}; q, t),$$

holds and implies

$$\begin{aligned} \mathfrak{H}_{\lambda/\mu}^{n,k}(\mathbb{Y}^{tq}, q, t) &= \langle P_\lambda(\mathbb{X}^\vee; q, t), \sum_{\rho} P_{\rho/\mu}(\mathbb{Y}; q, t) Q_\rho(\mathbb{X}; q, t) \rangle'_{q,t;n} \\ &= \langle P_\lambda(\mathbb{X}^\vee; q, t), Q_\lambda(\mathbb{X}; q, t) \rangle'_{q,t;n} Q_{\lambda/\mu}(\mathbb{Y}, q, t). \end{aligned} \quad (22)$$

Equality (12) ends the proof.  $\square$

### 3.2 Substitution dual formula

Setting  $\overline{\mathbb{Y}} = \{-y_1, \dots, -y_m, \dots\}$  if  $\mathbb{Y} = \{y_1, \dots, y_m, \dots\}$ <sup>2</sup>, one observes the following property.

**Theorem 3.2** *Let  $\mathbb{X} = \{x_1, \dots, x_n\}$  be an alphabet,  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition and  $\mu \subset \lambda$ . One has*

$$\mathfrak{H}_{\lambda/\mu}^{n,k}(-\overline{\mathbb{Y}}; q, t) = \mathfrak{H}_{\lambda'/\mu'}^{n,k}(\mathbb{Y}^{qt}; t, q) \quad (23)$$

where  $\mathbb{Y}^{qt} = \frac{1-q}{1-t} \mathbb{Y}$ .

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<sup>2</sup>The operation  $\mathbb{Y} \rightarrow \overline{\mathbb{Y}}$  makes sense for virtual alphabet since it sends any homogeneous symmetric polynomial  $P(\mathbb{Y})$  of degree  $p$  to  $(-1)^p P(\mathbb{Y})$ .

**Proof** It suffices to show that

$$\mathfrak{H}_{\lambda/\mu}^{n,k}(-\overline{\mathbb{Y}}; q, t) = \frac{1}{n!} \prod_{(i,j) \in \lambda} \frac{1 - q^{i-1}t^{n-j+1}}{1 - q^i t^{n-j}} \text{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{\lambda'/\mu'}(\mathbb{Y}, t, q).$$

The proof of this identity is almost the same than the proof of (20) except than one uses the formula

$$\prod (1 + x_i y_j) Q_\mu(\mathbb{X}; t, q) = \sum_{\rho} Q_\rho(\mathbb{X}; q, t) Q_{\rho'/\mu'}(\mathbb{Y}; t, q),$$

which is a special case of identity (16).  $\square$

Note that in the case of partitions, one has

**Corollary 3.3**

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}}, q, t) = \frac{1}{n!} \prod_{(i,j) \in \lambda} \frac{1 - q^{i-1}t^{n-j+1}}{1 - q^i t^{n-j}} \text{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{\lambda'}(\mathbb{Y}, t, q) \quad (24)$$

**Example 3.4** Consider the following equality

$$\mathfrak{H}_{41/3}^{2,3}(-\overline{\mathbb{Y}}; q, t) = (*) \text{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{2111/111}(\mathbb{Y}; t, q).$$

where  $\mathbb{X} = \{x_1, x_2\}$ . The coefficient  $(*)$  is computed as follows. One writes the partition  $[41]$  in a rectangle of height 2 and length 4.

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Each  $\times$  of coordinate  $(i, j)$  is read as the fraction  $[i, j] := \frac{1 - q^{i-1}t^{3-j}}{1 - q^i t^{2-j}}$ . Hence

$$(*) = [1, 2][1, 1][2, 1][3, 1][4, 1] = \frac{(1-t)(1-t^2)(1-qt^2)(1-q^2t^2)(1-q^3t^2)}{(1-q)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)}$$



## 4 A formula relying the polynomials $P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right)$ and $P_\lambda (\mathbb{X}; q, q^k)$

When  $t = q^k$  with  $k \in \mathbb{N}$ , Corollary 3.3 gives

**Corollary 4.1**

$$\mathfrak{H}_\lambda^{n,k}(-\overline{\mathbb{Y}}, q, q^k) = \beta_\lambda^{n,k}(q) Q_{\lambda'}(\mathbb{Y}; q^k, q). \quad (25)$$

where

$$\beta_\lambda^{n,k}(q) = \prod_{i=0}^{n-1} \begin{bmatrix} \lambda_{n-i} - 1 + k(i+1) \\ k-1 \end{bmatrix}_q$$

and  $\begin{bmatrix} n \\ p \end{bmatrix}_q = \frac{(1-q^n) \dots (1-q^{n-p+1})}{(1-q) \dots (1-q^p)}$  denotes the  $q$ -binomial.

**Proof** From Corollary 3.3, it remains to compute  $\text{C.T.}\{\Delta(\mathbb{X}, q, t)\}$ . The evaluation of this term is deduced from the  $q$ -Dyson conjecture <sup>3</sup>

$$\text{C.T.}\{\Delta(x; q, q^k)\} = n! \prod_{i=1}^n \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}_q,$$

and can be found in [11] examples 1 p 372.

Hence,

$$\mathfrak{H}_\lambda^{n,k}(-\overline{\mathbb{Y}}, q, q^k) = \beta_\lambda^{n,k}(q) Q_{\lambda'}(\mathbb{Y}, q^k, q),$$

where

$$\beta_\lambda^{n,k}(q) = \prod_{(i,j) \in \lambda} \frac{1 - q^{i+k(n-j+1)-1}}{1 - q^{i+k(n-j)}} \prod_{i=1}^n \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}_q. \quad (26)$$

But,

$$\prod_{(i,j) \in \lambda} \frac{1 - q^{i+k(n-j+1)-1}}{1 - q^{i+k(n-j)}} = \prod_{i=0}^{n-1} \prod_{j=1}^{\lambda_{n-i}} \frac{1 - q^{j+k(i+1)-1}}{1 - q^{j+ki}}.$$

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<sup>3</sup>see [14] for a proof.

Hence, rearranging the factors appearing in the right hand side of Equality (26), one obtains

$$\begin{aligned}\beta_\lambda^{n,k}(q) &= \prod_{i=0}^{n-1} \left( \left[ \begin{matrix} (i+1)k-1 \\ ik \end{matrix} \right]_q \prod_{j=1}^{\lambda_{n-i}} \frac{1-q^{j+k(i+1)-1}}{1-q^{j+ki}} \right) \\ &= \prod_{i=0}^{n-1} \left[ \begin{matrix} \lambda_{n-i}-1+k(i+1) \\ k-1 \end{matrix} \right]_q.\end{aligned}\tag{27}$$

This ends the proof.  $\square$

**Example 4.2** Set  $k = 2, n = 3$  and consider the polynomial

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}}; q, q^2) = \frac{1}{n!} \int_{-\overline{\mathbb{Y}}} P_{[32]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (1 - x_i x_j^{-1}).$$

One has,

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}}; q, q^2) = \frac{(1-q^5)(1-q^8)}{(1-q)^2} Q_{[221]}(\mathbb{Y}; q^2, q).$$

Let

$$\Omega_S := \frac{1}{n!} \int_{\mathbb{X}} \prod_{i \neq j} (1 - x_i x_j^{-1}) \tag{28}$$

and for each  $v \in \mathbb{Z}^n$ ,

$$\tilde{S}_v(\mathbb{X}) = \det \left( x_i^{v_j+n-j} \right) \prod_{i < j} (x_i - x_j)^{-1}.$$

**Lemma 4.3** *If  $v$  is any vector in  $\mathbb{Z}^n$ , one has*

$$\Omega_S \tilde{S}_v(\mathbb{X}) = S_v(\mathbb{X}) := \det(S^{v_i-i+j}(\mathbb{X})) \tag{29}$$

**Proof** The identity is obtain by the direct computation:

$$\begin{aligned}\frac{1}{n!} \int_{\mathbb{X}} \tilde{S}_v(\mathbb{X}) \prod_{i < j} (1 - x_i x_j^{-1}) &= \frac{1}{n!} \int_{\mathbb{X}} \det \left( x_i^{v_j-j+1} \right) \det(x_i^{j-1}) \\ &= \frac{1}{n!} \int_{\mathbb{X}} \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}_n} \text{sign}(\sigma_1 \sigma_2) \prod_i x^{v_{\sigma_1(i)} - \sigma_1(i) + \sigma_2(i) - 1} \\ &= \frac{1}{n!} \sum_{\sigma_1 \sigma_2} \text{sign}(\sigma_1 \sigma_2) \prod_i S_{v_{\sigma_1(i)} - \sigma_1(i) + \sigma_2(j)} \\ &= \det(S^{v_i-i+j}(\mathbb{X})).\end{aligned}$$

□

In particular,  $\Omega_S$  lets invariant any symmetric polynomial. The operator

$$\mathfrak{A}_m := \Omega_S \Lambda^n(\mathbb{X})^{-m} \quad (30)$$

acts on symmetric polynomials by subtracting  $m$  on each part of partitions appearing in their expansion in the Schur basis.

**Example 4.4** If  $\mathbb{X} = \{x_1, x_2, x_3\}$  consider the polynomial, and  $\lambda = [320]$ . One has

$$P_{32}(\mathbb{X}; q, t) = S_{32}(\mathbb{X}) + \frac{(-q+t) S_{311}(\mathbb{X})}{qt-1} + \frac{(q+1)(qt^2-1)(-q+t) S_{221}(\mathbb{X})}{(qt-1)^2(qt+1)}.$$

Hence,

$$\begin{aligned} \mathfrak{A}_1 P_{32}(\mathbb{X}; q, t) &= \frac{(-q+t) S_2(\mathbb{X})}{qt-1} + \frac{(q+1)(qt^2-1)(-q+t) S_{11}(\mathbb{X})}{(qt-1)^2(qt+1)} \\ &= \frac{(-q+t)(t+1)(q^2t-1) P_{11}(\mathbb{X}; q, t)}{(qt-1)^2(qt+1)} + \frac{(-q+t) P_2(\mathbb{X}; q, t)}{qt-1}. \end{aligned}$$

**Theorem 4.5** If  $\lambda$  denotes a partition of length at most  $n$ , one has

$$\mathfrak{A}_{(k-1)(n-1)} P_\lambda(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right) \quad (31)$$

**Proof** From the definitions of the operators  $\mathfrak{A}_m$  (30) and  $\Omega_S$  (28), one obtains

$$\mathfrak{A}_{(k-1)(n-1)} P_\lambda(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \frac{1}{n!} \int_{\mathbb{X}} P_\lambda(\mathbb{X}; q, q^k) \Delta(\mathbb{X}, q, q^k).$$

Corollary 4.1 implies

$$\begin{aligned} \frac{1}{n!} \int_{\mathbb{X}} P_\lambda(\mathbb{X}; q, q^k) \Delta(\mathbb{X}, q, q^k) &= \mathfrak{H}_\lambda^{n,k}(\mathbb{X}; q, q^k) \\ &= \beta_\lambda^{n,k}(q) Q_{\lambda'}(-\overline{\mathbb{X}}; q^k, q) \end{aligned}$$

But, from Lemma 2.1, one has

$$Q_{\lambda'}(-\overline{\mathbb{X}}; q^k, q) = (-1)^{|\lambda|} Q_{\lambda'}(-\mathbb{X}; q^k, q) = P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right).$$

The result follows. □

**Example 4.6** Set  $k = 2$ ,  $n = 3$  and  $\lambda = [2]$ . One has

$$P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - qx_j) = -q^3 S_{[6,2]} + q^2 \frac{q^3 - 1}{q - 1} S_{[6,1,1]} \\ + \frac{q^2(q^5 - 1)}{q^3 - 1} S_{[5,3]} - \frac{q(q^2 + 1)(q^5 - 1)}{q^3 - 1} S_{[5,2,1]} - \frac{q(q^7 - 1)}{q^3 - 1} S_{[4,3,1]} + \frac{q^7 - 1}{q - 1} S_{[4,2,2]}.$$

And,

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - qx_j) = \frac{q^7 - 1}{q - 1} S_{[2]}.$$

Since,

$$P_{[2]} \left( \frac{x_1 + x_2 + x_3}{1 + q}; q, q^2 \right) = \frac{1 - q}{1 - q^3} S_{[2]}$$

one obtains

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - qx_j) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 7 \\ 1 \end{bmatrix}_q P_{[2]} \left( \frac{x_1 + x_2 + x_3}{1 + q}; q, q^2 \right).$$

As a consequence, one has

**Corollary 4.7** If  $\lambda = \mu + [((k - 1)(n - 1))^n]$ ,

$$P_\mu(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_\lambda \left( \frac{1 - q}{1 - q^k} \mathbb{X}; q, q^k \right).$$

**Proof** Since the size of  $\mathbb{X}$  is  $n$ ,

$$P_\lambda(\mathbb{X}; q, q^k) = P_\mu(\mathbb{X}; q, q^k) (x_1 \dots x_n)^{(k-1)(n-1)}.$$

Then, the result is a direct consequence of Theorem 4.5.  $\square$

**Example 4.8** Set  $k = 3$ ,  $n = 2$  and  $\lambda = [5, 2]$ . One has,

$$P_{[5,2]}(x_1 + x_2; q, q^3) (x_1 - qx_2) (x_1 - q^2 x_2) (x_2 - qx_1) (x_2 - q^2 x_1) = \\ q^3 S_{[9,2]} + \frac{(1 - q^7)(1 + q^4)}{1 - q^5} S_{[7,4]} - \frac{(1 - q^2)(1 + q)(1 + q^2)(1 + q^4)}{1 - q^5} S_{[8,3]}.$$

This implies

$$\mathfrak{A}_2 P_{[5,2]}(x_1 + x_2; q, q^3) (x_1 - qx_2) (x_1 - q^2 x_2) (x_2 - qx_1) (x_2 - q^2 x_1) = \\ (x_1 x_2)^{-2} P_{[5,2]}(x_1 + x_2; q, q^3) (x_1 - qx_2) (x_1 - q^2 x_2) (x_2 - qx_1) (x_2 - q^2 x_1) = \\ P_{[3]}(x_1 + x_2; q, q^3) (x_1 - qx_2) (x_1 - q^2 x_2) (x_2 - qx_1) (x_2 - q^2 x_1).$$

One verifies that

$$P_{[3]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2x_2)(x_2 - qx_1)(x_2 - q^2x_1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \begin{bmatrix} 10 \\ 2 \end{bmatrix}_q P_{[5,2]}(\frac{x_1 + x_2}{1 + q + q^2}; q, q^3).$$

**Remark 4.9** If  $\mu$  is the empty partition, Corollary 4.7 gives

$$\prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_{\lambda}^{n,k}(q) P_{[(k-1)(n-1)]^n} \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right).$$

This equality generalizes an identity given in [1]:

$$\prod_{i < j} (x_i - x_j)^{2(k-1)} = \frac{(-1)^{\frac{(k-1)n(n-1)}{2}}}{n!} \binom{kn}{k, \dots, k} P_{n^{(n-1)(k-1)}}^{(k)}(-\mathbb{X}),$$

where  $P_{\lambda}^{(k)}(\mathbb{X}) = \lim_{q \rightarrow 1} P_{\lambda}^{(\alpha)}(\mathbb{X}; q, q^k)$  denotes a Jack polynomial (see *e.g.* [11]).

The expansion of the powers of the discriminant and their  $q$ -deformations in different basis of symmetric functions is a difficult problem having many applications, for example, in the study of Hua-type integrals (see *e.g.* [4, 6]) or in the context of the factional quantum Hall effect (*e.g.* [3, 5, 7, 13]).

Note that in [2], we gave an expression of an other  $q$ -deformation of the powers of the discriminant as staircase Macdonald polynomials. This deformation is also relevant in the study of the expansion of  $\prod_{i < j} (x_i - x_j)^{2k}$  in the Schur basis, since we generalized [2] a result of [5].

## 5 Macdonald polynomials at $t = q^k$ as eigenfunctions

Let  $\mathbb{Y} = \{y_1, \dots, y_{kn}\}$  be an alphabet of cardinality  $kn$  with  $y_1 = x_1, \dots, y_n = x_n$ . One considers the symmetrizer  $\pi_{\omega}$  defined by

$$\pi_{\omega} f(y_1, \dots, y_{kn}) = \prod_{i < j} (x_i - x_j)^{-1} \sum_{\sigma \in \mathfrak{S}_{kn}} \text{sign}(\sigma) f(y_{\sigma(1)}, \dots, y_{\sigma(kn)}) y_{\sigma(1)}^{kn-1} \dots y_{\sigma(kn-1)}.$$

Note that  $\pi_{\omega}$  is the isobaric divided difference associated to the maximal permutation  $\omega$  in  $\mathfrak{S}_{kn}$ .

This operator applied to a symmetric function of the alphabet  $\mathbb{X}$  increases the alphabet from  $\mathbb{X}$  to  $\mathbb{Y}$  in its expansion in the Schur basis, since

$$\pi_\omega S_\lambda(\mathbb{X}) = S_\lambda(\mathbb{Y}). \quad (32)$$

Indeed, the image of the monomial  $y_1^{i_1} \dots y_{kn}^{i_{kn}}$  is the Schur function  $S_I(\mathbb{Y})$ . Since

$$\pi_\omega S_\lambda(\mathbb{X}) = \pi_\omega x_1^{\lambda_1} \dots x_n^{\lambda_n} = \pi_\omega y_1^{\lambda_1} \dots y_n^{\lambda_n} y_{n+1}^0 \dots y_{kn}^0,$$

one recovers Equality (32).

One defines the operator  $\pi^{tq}$  which consists in applying  $\pi_\omega$  and specializing the result to the alphabet

$$\mathbb{X}^{tq} := \{x_1, \dots, x_n, qx_1, \dots, qx_n, \dots, q^{k-1}x_1, \dots, q^{k-1}x_n\}.$$

From Equality (32), one has

$$\pi_\omega^{tq} S_\lambda(\mathbb{X}) = S_\lambda((1 + q + \dots + q^{k-1})\mathbb{X}), \quad (33)$$

for  $l(\lambda) \leq n$ . Furthermore, the expansion of  $S_\lambda((1 + q + \dots + q^{k-1})\mathbb{X})$  in the Schur basis being triangular, the operator  $\pi^{tq}$  defines an automorphism of the space  $Sym_{\leq n}$  generated by the Schur functions indexed by partitions whose length are less or equal to  $n$ , *i.e.* for each function  $f \in Sym_{\leq n}$ , one has

$$\pi^{tq} f(\mathbb{X}) = f(\mathbb{X}^{tq}). \quad (34)$$

In particular, one has

**Lemma 5.1** *Let  $\lambda$  be a partition such that  $l(\lambda) \leq n$  then*

$$\pi_\omega^{tq} P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, t = q^k \right) = P_\lambda(\mathbb{X}, q, q^k). \quad (35)$$

**Proof** It suffices to remark that  $P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right) \in Sym_{\leq n}(\mathbb{X})$ .<sup>4</sup>

It follows from (34),

$$\pi_\omega^{tq} P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right) = P_\lambda \left( \frac{1-q}{1-q^k} \mathbb{X}^{tq}; q, q^k \right) = P_\lambda(\mathbb{X}, q, q^k).$$

---

<sup>4</sup>This can be seen as a consequence of the determinantal expression of the expansion of  $P_\lambda(\mathbb{X}, q, t)$  in the Schur basis evaluated on the alphabet  $\mathbb{X}^{tq}$  (see [8]).

□

Consider the operator  $\mathfrak{M} : f \rightarrow \mathfrak{M}f$  defined by

$$\mathfrak{M} := (x_1 \dots x_n)^{(k-1)(1-n)} \pi_{\omega}^{tq} \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j).$$

The following theorem shows that the Macdonald polynomials are the eigenfunctions of the operator  $\mathfrak{M}$ .

**Theorem 5.2** *The Macdonald polynomials  $P_{\lambda}(\mathbb{X}; q, q^k)$  are eigenfunctions of  $\mathfrak{M}$ . The eigenvalue associated to  $P_{\mu}(\mathbb{X}; q, q^k)$  is  $\beta_{\mu + ((k-1)(n-1))^n}^{n,k}(q)$ . Furthermore, if  $k > 1$ , the dimension of each eigenspace is 1.*

**Proof** From Corollary 4.7, one has

$$P_{\mu}(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_{\lambda}^{n,k}(q) P_{\lambda} \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right)$$

where  $\lambda = \mu + ((k-1)(n-1))^n$ . Applying  $\pi_{\omega}^{tq}$  to the left and the right hand sides of this equality, one obtains from Lemma 5.1

$$\begin{aligned} \pi_{\omega}^{tq} P_{\mu}(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) &= \beta_{\lambda}^{n,k}(q) \pi_{\omega}^{tq} P_{\lambda} \left( \frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right) \\ &= \beta_{\lambda}^{n,k}(q) P_{\lambda}(\mathbb{X}; q, q^k). \end{aligned}$$

Since the cardinality of  $\mathbb{X}$  is  $n$ , one has

$$P_{\lambda}(\mathbb{X}; q, q^k) = (x_1 \dots x_n)^{(k-1)(n-1)} P_{\lambda}(\mathbb{X}; q, q^k),$$

and

$$\mathfrak{M} P_{\mu}(\mathbb{X}; q, q^k) = \beta_{\mu + ((k-1)(n-1))^n}^{n,k}(q) P_{\mu}(\mathbb{X}; q, q^k). \quad (36)$$

Suppose now that  $k > 1$ . It remains to prove that the dimensions of the eigenspaces equal 1. More precisely, It suffices to show that  $\beta_{\lambda}(q) = \beta_{\mu}(q)$  implies  $\lambda = \mu$ . The denominators of  $\beta_{\lambda}(q)$  and  $\beta_{\mu}(q)$  being the same, one needs only to examine the numerators, that is the products  $\gamma_{\lambda} = \prod_{i=0}^{n-1} (q^{\lambda_{n-i} + ki}; q)_{k-1}$  and  $\gamma_{\mu} = \prod_{i=0}^{n-1} (q^{\mu_{n-i} + ki}; q)_{k-1}$ . One needs the following lemma.

**Lemma 5.3** *Let  $I = \{i_1, \dots, i_n\}$  and  $J = \{j_1, \dots, j_m\}$  be two finite subsets of  $\mathbb{N} \setminus \{0\}$ . Then,  $I \neq J$  implies  $\prod_{i \in I} (1 - q^i) \neq \prod_{j \in J} (1 - q^j)$ .*

**Proof** Without loss of generalities, one can suppose  $I \cap J = \emptyset$ . Suppose that  $i_1 \leq \dots \leq i_n$  and  $j_1 \leq \dots \leq j_m$ . Then, expanding the two products, one finds

$$\prod_{i \in I} (1 - q^i) = 1 - q^{i_1} + \sum_{l > i_1} (*) q^l \neq 1 - q^{j_1} + \sum_{l > j_1} (*) q^l = \prod_{j \in J} (1 - q^j).$$

□

Each term  $(q^{\lambda_{n-i} + ki}; q)_{k-1}$  is characterized by the degree of its factor of lower degree :  $\lambda_{n-i} + ki$ . Hence, from Lemma 5.3,  $\beta_\lambda(q) = \beta_\mu(q)$  implies that it exists a permutation  $\sigma$  of  $\mathfrak{S}_n$  verifying

$$\lambda_i + k(n - i) = \mu_{\sigma_i} + k(n - \sigma_i),$$

for each  $i$ . But, since  $\lambda$  is decreasing, one has

$$\lambda_i + k(n - i) - \lambda_{i-1} - k(n - i + 1) \leq 0.$$

And then,

$$\mu_{\sigma_i} + k(n - \sigma_i) - \mu_{\sigma_{i-1}} - k(n - \sigma_{i-1}) \leq 0. \quad (37)$$

But, since  $\mu$  is decreasing,  $\sigma_{i-1} - \sigma_i$  has the same sign than  $\mu_{\sigma_i} - \mu_{\sigma_{i-1}}$ . As a consequence, Inequality (37) implies  $\sigma_i > \sigma_{i-1}$  for each  $i$ . The only possibility is  $\sigma = Id$ , which ends the proof. □

**Example 5.4** If  $n = 5$ , the eigenvalues associated to the partitions of 4 are

$$\begin{aligned} \beta_{[4k, 4k-4, 4k-4, 4k-4, 4k-4]}^{4,k} &= \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-1 \\ k-1 \end{bmatrix}_q, \\ \beta_{[4k-1, 4k-3, 4k-4, 4k-4, 4k-4]}^{4,k} &= \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-2 \\ k-1 \end{bmatrix}_q, \\ \beta_{[4k-2, 4k-2, 4k-4, 4k-4, 4k-4]}^{4,k} &= \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-3 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-3 \\ k-1 \end{bmatrix}_q, \\ \beta_{[4k-2, 4k-3, 4k-3, 4k-4, 4k-4]}^{4,k} &= \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-3 \\ k-1 \end{bmatrix}_q, \\ \beta_{[4k-3, 4k-3, 4k-3, 4k-3, 4k-4]}^{4,k} &= \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-4 \\ k-1 \end{bmatrix}_q. \end{aligned}$$

## 6 Expression of $\mathfrak{M}$ in terms of Cherednik elements

In this paragraph, we restate Proposition 5.2 in terms of Cherednik operators. Cherednik's operators  $\{\xi_i; i \in \{1, \dots, n\}\} =: \Xi$  are commutative elements



of the double affine Hecke algebra. The Macdonald polynomials  $P_\lambda(\mathbb{X}; q, t)$  are eigenfunctions of symmetric polynomials  $f(\Xi)$  and the eigenvalues are obtained substituting each occurrence of  $\xi_i$  in  $f(\Xi)$  by  $q^{\lambda_i} t^{n-i}$  (see [10] for more details).

Suppose that  $k > 1$  and consider the operator  $\tilde{\mathfrak{M}} : f \rightarrow \tilde{\mathfrak{M}}f$  defined by

$$\tilde{\mathfrak{M}} := \prod_{i=1}^{k-1} (1 - q^i)^n \mathfrak{M}. \quad (38)$$

From Proposition 5.2, one has

$$\tilde{\mathfrak{M}}P_\lambda(\mathbb{X}; q, q^k) = \prod_{i=0}^{n-1} \prod_{j=1}^{k-1} (1 - q^{\lambda_{n-i} + k(i+1) - j}) P_\lambda(\mathbb{X}; q, q^k). \quad (39)$$

The following proposition shows that  $\tilde{\mathfrak{M}}$  admits a closed expression in terms of Cherednick elements.

**Proposition 6.1** *One supposes that  $k > 1$ . For any symmetric function  $f$ , one has*

$$\tilde{\mathfrak{M}}f(\mathbb{X}) = \prod_{l=1}^{k-1} \prod_{i=1}^n (1 - q^{l+k} \xi_i) f(\mathbb{X}). \quad (40)$$

**Proof** From Theorem 5.2, it suffices to prove the formula (40) for  $f = P_\lambda$ . The polynomial  $P_\lambda(\mathbb{X}; q, t)$  is an eigenfunction of the operator  $\prod_{l=1}^{k-1} \prod_{i=1}^n (1 - q^{l+k} \xi_i)$  and its eigenvalues is  $\prod_{l=1}^{k-1} \prod_{i=1}^n (1 - q^{l+\lambda_i} t^{n-i+1}) P_\lambda(\mathbb{X}; q, t)$ . Hence, setting  $t = q^k$ , we obtain

$$\prod_{l=1}^{k-1} \prod_{i=1}^n (1 - q^{l+k} \xi_i) P_\lambda(\mathbb{X}; q, q^k) = \prod_{l=1}^{k-1} \prod_{i=1}^n (1 - q^{l+\lambda_i + k(n-i+1)}) P_\lambda(\mathbb{X}; q, q^k).$$

Comparing this expression to Equality (38), one finds the result.  $\square$

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